Packing, Counting and Covering Hamilton cycles in random directed graphs

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Abstract

A Hamilton cycle in a digraph is a cycle that passes through all the vertices, where all the arcs are oriented in the same direction. The problem of finding Hamilton cycles in directed graphs is well studied and is known to be hard. One of the main reasons for this, is that there is no general tool for finding Hamilton cycles in directed graphs comparable to the so called Posá 'rotation-extension' technique for the undirected analogue. Let $\mathcal{D}(n,p)$ denote the random digraph on vertex set $[n]$, obtained by adding each directed edge independently with probability p. Here we present a general and a very simple method, using known results, to attack problems of packing and counting Hamilton cycles in random directed graphs, for every edge-probability $p > \log^C(n)/n$. Our results are asymptotically optimal with respect to all parameters and apply equally well to the undirected case.

1 Introduction

A *Hamilton cycle* in a graph or a directed graph is a cycle passing through every vertex of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity is one of the most central notions in graph theory, and has been intensively studied by numerous researchers in the last couple of decades.

The decision problem of whether a given graph contains a Hamilton cycle is known to be $\mathcal{NP}$-hard and is one of Karp's list of 21 $\mathcal{NP}$-hard problems [23]. Therefore, it is important to find general sufficient conditions for Hamiltonicity and indeed, many interesting results were obtained in this direction.

Once Hamiltonicity has been established for a graph there are many questions of further interest. For example, the following are natural questions:

- Let G be a graph with minimum degree $\delta(G)$. Is it possible to find roughly $\delta(G)/2$ edge-disjoint Hamilton cycles? (This problem is referred to as the *packing* problem.)

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- Let $\Delta(G)$ denote the maximum degree of G. Is it possible to find roughly $\Delta(G)/2$ Hamilton cycles for which every edge $e \in E(G)$ appears in at least one of these cycles? (This problem is referred to as the *covering* problem.)

- How many distinct Hamilton cycles does a given graph have? (This problem is referred to as the *counting* problem.)

All of the above questions have a long history and many results are known. Let us define $\mathcal{G}(n, p)$ to be the probability space of graphs on a vertex set $[n] := \{1, \ldots, n\}$, such that each possible (unordered) pair xy of elements of $[n]$ appears as an edge independently with probability p. We say that a graph $G \sim \mathcal{G}(n, p)$ satisfies a property $\mathcal{P}$ of graphs with high probability (w.h.p.) if the probability that G satisfies $\mathcal{P}$ tends to 1 as n tends to infinity.

Packing. The question of packing in the probabilistic setting was firstly discussed by Bollobás and Frieze in the 80's. They showed in [4] that if $\{G_i\}_{i=0}^{\binom{n}{2}}$ is a random graph process on $[n]$, where G_0 is the empty graph and G_i is obtained from G_{i-1} by adjoining a non-edge of G_{i-1} uniformly at random, as soon as G_i has minimum degree k (where k is a fixed integer), it has $\lfloor k/2 \rfloor$ edge-disjoint Hamilton cycles plus a disjoint perfect matching if k is odd. This result generalizes an earlier result of Bollobás [3] who proved (among other things) that for $p = \frac{\ln n + \ln \ln n + \omega(1)}{n}$, a typical graph $G \sim \mathcal{G}(n, p)$ is Hamiltonian. Note that this value of p is optimal in the sense that for $p = \frac{\ln n + \ln \ln n - \omega(1)}{n}$, it is known that w.h.p. a graph $G \sim \mathcal{G}(n, p)$ satisfies $\delta(G) \leq 1$, and therefore is not Hamiltonian. Later on, Frieze and Krivelevich showed in [14] that for $p = (1 + o(1))\frac{\ln n}{n}$, a graph $G \sim \mathcal{G}(n, p)$ w.h.p. contains $\lfloor \delta(G)/2 \rfloor$ edge-disjoint Hamilton cycles (in fact, this was proven only using pseudo-random hypothesis), which has afterwards been improved by Ben-Shimon, Krivelevich and Sudakov in [2] to $p \leq 1.02\frac{\ln n}{n}$. We remark that in this regime of p, w.h.p. $G \sim \mathcal{G}(n, p)$ is quite far from being regular. As the culmination of a long line of research Knox, Kühn and Osthus [24], Krivelevich and Samotij [26] and Kühn and Osthus [28] completely solved this question for the entire range of p.

For the non-random case, it is worth mentioning a recent remarkable result due to Csaba, Kühn, Lo, Osthus and Treglown [5] which proved that for large enough n and $d \geq \lfloor n/2 \rfloor$, every d-regular graph on n vertices contains $\lfloor d/2 \rfloor$ edge-disjoint Hamilton cycles and one disjoint perfect matching in case d is odd. This result settles a long standing problem due to Nash-Williams [31] for large graphs.

Covering. The problem of covering the edges of a random graph was firstly studied in [18] by Glebov, Krivelevich and Szabó. It is shown that for $p \geq n^{-1+\varepsilon}$, the edges of a typical $G \sim \mathcal{G}(n, p)$ can be covered by $(1 + o(1))np/2$ edge-disjoint Hamilton cycles. Furthermore they proved analogous results also in the pseudo-random setting. In [19], Hefetz, Lapinskas, Kühn and Osthus improved it by showing that for some $C > 0$ and $\frac{\log^C(n)}{n} \leq p \leq 1 - n^{-1/8}$, one can cover all the edges of a typical graph $G \sim \mathcal{G}(n, p)$ with $\lceil \Delta(G)/2 \rceil$ Hamilton cycles.

Counting. Given a graph G, let $h(G)$ denote the number of distinct Hamilton cycles in G. Strengthening the classical theorem of Dirac from the 50's [8], Sárközy, Selkow and Szemerédi [33] proved that every graph G on n vertices with minimum degree at least $n/2$ contains not only one but at least $c^n n!$ Hamilton cycles for some small positive constant c. They also conjectured that this c could be improved to $1/2 - o(1)$. This was later proven by Cuckler and Kahn [7]. In fact, Cuckler and Kahn proved a stronger result: every graph G on n vertices with minimum degree $\delta(G) \geq n/2$

has $h(G) \geq \left(\frac{\delta(G)}{e}\right)^n (1 - o(1))^n$. A typical random graph $G \sim \mathcal{G}(n, p)$ with $p > 1/2$ shows that this estimate is sharp (up to the $(1 - o(1))^n$ factor). Indeed, in this case with high probability $\delta(G) = pn + o(n)$ and the expected number of Hamilton cycles is $p^n(n - 1)! < (pn/e)^n$.

In the random/pseudo-random setting, building on ideas of Krivelevich [25], in [17] Glebov and Krivelevich showed that for $p \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$ and for a typical $G \sim \mathcal{G}(n, p)$ we have $h(G) = (1 - o(1))^n n! p^n$. That is, the number of Hamilton cycles is, up to a sub-exponential factor, concentrated around its mean. For larger values of p, Janson showed [21] that the distribution of $h(G)$ is log-normal, for $G \sim \mathcal{G}(n, p)$ with $p = \omega(n^{-1/2})$.

In this paper we treat the three of these problems in the random directed setting. A *directed graph* (or *digraph*) is a pair $D = (V, E)$ with a set of *vertices* V and a set of *arcs* E, where each arc is an ordered pair of elements of V. A directed graph is called *oriented*, if for every pair of vertices $u, v \in V$, at most one of the directed edges $\overrightarrow{uv}$ or $\overrightarrow{vu}$ appears in the graph. A *tournament* is an oriented complete graph. A *Hamilton cycle* in a digraph is a cycle going through all the vertices exactly once, where all the arcs are oriented in the same direction in a cyclic order. Given a directed graph D and a vertex $v \in V$, we let $d_D^+(v)$ and $d_D^-(v)$ denote its out- and in- degree in D.

Let $\mathcal{D}(n, p)$ be the probability space consisting of all directed graphs on vertex set $[n]$ in which each possible arc is added with probability p independently at random. The problem of determining the range of values of p for which a typical graph $D \sim \mathcal{D}(n, p)$ is Hamiltonian goes back to the early 80's, where McDiarmid [30] showed, among other things, that an elegant coupling argument gives the inequality

$$\Pr[G \sim \mathcal{G}(n, p) \text{ is Hamiltonian}] \leq \Pr[D \sim \mathcal{D}(n, p) \text{ is Hamiltonian}].$$

Combined with the result of Bollobás [3] it follows that a typical $D \sim \mathcal{D}(n, p)$ is Hamiltonian for $p \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$. Later on, Frieze showed in [16] that the same conclusion holds for $p \geq \frac{\ln n + \omega(1)}{n}$. The result of Frieze is optimal in the sense that for $p = \frac{\ln n - \omega(1)}{n}$, it is not difficult to see that for a typical $D \sim \mathcal{D}(n, p)$ we have $\min_{v \in V}\{\delta^+(v), \delta^-(v)\} = 0$ and therefore D is not Hamiltonian. Robustness of Hamilton cycles in random digraphs was studied by Hefetz, Steger and Sudakov in [20] and by Ferber, Nenadov, Noever, Peter and Skorić in [13].

1.1 Our results

While in general/random/pseudo-random graphs there are many known results, much less is known about the problems of counting, packing and covering in the directed setting. The main difficulty is that in this setting the so called Posá rotation-extension technique (see [32]) does not work in its simplest form.

In this paper we present a simple method to attack and approximately solve all the above mentioned problems in random/pseudo-random directed graphs, with an optimal (up to a *polylog(n)* factor) density. Our method is also applicable in the undirected setting, and therefore reproves many of the above mentioned results in a simpler way.

The problem of packing Hamilton cycles in digraphs goes back to the 70's. Tilson [36] showed that every complete digraph has a Hamilton decomposition. Recently, a remarkable result of Kühn and Osthus (see [27]) proves that for any regular orientation of a sufficiently dense graph one can find a Hamilton decomposition. In the case of a random directed graph, not much is known regarding

packing Hamilton cycles. Our first result proves the existence of $(1 - o(1))np$ edge-disjoint Hamilton cycles in $\mathcal{D}(n, p)$.

Theorem 1.1. *For $p = \omega\left(\frac{\log^4 n}{n}\right)$, w.h.p. the digraph $D \sim \mathcal{D}(n, p)$ has $(1 - o(1))np$ edge-disjoint Hamilton cycles.*

We also show that in random directed graphs one can cover all the edges by not too many cycles.

Theorem 1.2. *Let $p = \omega\left(\frac{\log^2 n}{n}\right)$. Then, a digraph $D \sim \mathcal{D}(n, p)$ w.h.p. can be covered with $(1 + o(1))np$ directed Hamilton cycles.*

The problem of counting Hamilton cycles in digraphs was already studied in the early 70's by Wright in [38]. However, counting Hamilton cycles in tournaments is an even older problem which goes back to one of the first applications of the probabilistic method by Szele [34]. He proved that there are tournaments on n vertices with at least $(n-1)!/2^n$ Hamilton cycles. Thomassen [35] conjectured that in fact every regular tournament contains at least $n^{(1-o(1))n}$ Hamilton cycles. This conjecture was solved by Cuckler [6] who proved that every regular tournament on n vertices contains at least $\frac{n!}{(2+o(1))^n}$ Hamilton cycles. Ferber, Krivelevich and Sudakov [11] later extended Cuckler's result for every nearly cn-regular oriented graph for $c > 3/8$. Here, we count the number of Hamilton cycles in random directed graphs and improve a result of Frieze and Suen from [15]. We show that the number of directed Hamilton cycles in such random graphs is concentrated (up to a sub-exponential factor) around its mean.

Theorem 1.3. *Let $p = \omega\left(\frac{\log^2 n}{n}\right)$. Then, a digraph $D \sim \mathcal{D}(n, p)$ w.h.p. contains $(1 \pm o(1))^n n! p^n$ directed Hamilton cycles.*

Finally, the same proof method can be used to prove analogous results when instead working with pseudo-random graphs. We direct the reader to Definition 6.1 in Section 6.1 for the notion of pseudo-randomness used here. The following theorems show that at a cost of an additional polylog$n$ factor in the density we obtain analogues of Theorem 1.1, 1.2, 1.3 for pseudo-random digraphs. Below we will write $o_\lambda(1)$ for some quantity tending to 0 as $\lambda \to 0$.

Theorem 1.4. *Let D be a (n, λ, p) pseudo-random digraph where $p = \omega\left(\frac{\log^{14} n}{n}\right)$. Then D contains $(1 - o_\lambda(1))np$ edge-disjoint Hamilton cycles.*

Theorem 1.5. *Let D be a (n, λ, p) pseudo-random digraph where $p = \omega\left(\frac{\log^{14} n}{n}\right)$. Then D can be covered with $(1 + o_\lambda(1))np$ directed Hamilton cycles.*

Theorem 1.6. *Let D be a (n, λ, p) pseudo-random digraph where $p = \omega\left(\frac{\log^{14} n}{n}\right)$. Then D can be contains $(1 - o_\lambda(1))^n n! p^n$ directed Hamilton cycles.*

We have only included the proof of Theorem 1.4 which modifies the proof of Theorem 1.1 to the pseudo-random setting. The other results can be proven in a similar manner (these other proofs are in fact slightly easier).

Remark 1.7. *We also draw attention to the fact that all of our proofs also apply to $\mathcal{G}(n, p)$ with the same probability thresholds as in Theorem 1.1, 1.2 and 1.3. Although all these results are known in $\mathcal{G}(n, p)$ (and in fact even much more), our approach provides us with short and elegant proofs. For convenience, we state the exact statements which follow from our proofs:*

4

- *For $p = \omega\left(\frac{\log^4 n}{n}\right)$ our approach gives that $G \sim \mathcal{G}(n, p)$ whp contains $(1 - o(1))np/2$ edge disjoint Hamitlon cycles. As mentioned in the packing section above, here it is known that for all p whp $G \sim \mathcal{G}(n, p)$ contains $\lfloor \delta(G)/2 \rfloor$ edge disjoint Hamilton cycles (see [24], [26] and [28]).*

- *For $p = \omega\left(\frac{\log^2 n}{n}\right)$ our approach gives that $G \sim \mathcal{G}(n, p)$ whp contains $(1 + o(1))np/2$ Hamilton cycles covering all edges of G. As mentioned in the covering section above, here it is known that there is some constant $C > 0$ such that for $\frac{\log^C n}{n} \le p \le 1 - n^{-1/18}$ whp $G \sim \mathcal{G}(n, p)$ has an edge covering with $\lceil \Delta(G)/2 \rceil$ Hamilton cycles (see [19]).*

- *For $p = \omega\left(\frac{\log^2 n}{n}\right)$ our approach gives that $G \sim \mathcal{G}(n, p)$ whp contains $(1 \pm o(1))^n n! p^n$ Hamilton cycles. As mentioned in the counting section above, here it is known that such a bound already applies for $p > \frac{\log n + \log \log n + \omega(1)}{n}$.*

1.2 Notation and terminology

We denote by D_n the *complete* directed graph on n vertices (that is, all the possible $n(n-1)$ arcs appear), and by $D_{n,m}$ the complete bipartite digraph with parts $[n]$ and $[m]$. Given a directed graph F and a vector $\bar{p} \in (0, 1]^{E(F)}$, we let $\mathcal{D}(F, \bar{p})$ denote the probability space of sub-digraphs D of F, where for each arc $e \in E(F)$, we add e into $E(D)$ with probability p_e, independently at random. In the special case where $p_e = p$ for all e, we simply denote it by $\mathcal{D}(F, p)$. In the case where $F = D_n$, we write $\mathcal{D}(n, p)$ and in the case $F = D_{n,m}$ we write $\mathcal{D}(n, m, p)$. Given a digraph D and two sets $X, Y \subset V(D)$ we write $E_D(X, Y) = \{\overrightarrow{xy} \in E(D) : x \in X, y \in Y\}$. Also let $e_D(X, Y) = |E_D(X, Y)|$ and $e_D(X) = |E_D(X, X)|$. We will also occasionally make use of the same notation for graphs G, i.e. $e_G(X, Y)$. For a vertex v we denote $N_D^+(v) = E_D(\{v\}, V(D))$ and $N_D^-(v) = E_D(V(D), \{v\})$. Let $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$. Lastly, we write $x \in a \pm b$ to mean that x is in the interval $[a - b, a + b]$.

2 Overview and auxiliary results

2.1 Proof overview

Our aim in this subsection is to provide an overview of the proofs of Theorems 1.1, 1.2 and 1.3. In particular, we hope to highlight the similarities and differences which occur for the packing, counting and covering problems. To do this, we will first describe an approach to solve similar problems for a more restricted model of random digraph. We then outline how these results can be used to solve the corresponding problems for $\mathcal{D}(n, p)$.

Suppose that we are given a partition $[n] = V_0 \cup V_1 \cup \cdots \cup V_\ell$, with $|V_0| = s$ and $|V_j| = m$ for all $j \in [\ell]$ so that $n = m\ell + s$ (here $s = \omega(m)$ and $\ell = \text{polylog}(n)$). Consider the following way to select random digraph F:

1. For all $j \in [\ell - 1]$, directed edges from V_j and V_{j+1} are adjoined to F with probability p_{in} independently. Let F_j denote this sub-digraph of F;

2. The directed edges (a) in V_0 (b) from V_0 to V_1 (c) from V_ℓ to V_0 and (d) from V_ℓ to V_1 are adjoined to F with probability p_{ex} independently. Let F_0 denote this subdigraph of F.

This selection process gives a distribution on a set of digraphs. We will write $\mathcal{F}$ to denote this distribution, and write $F \sim \mathcal{F}$ to denote a digraph F chosen according to it. We will describe how to show that if $F \sim \mathcal{F}$ then whp, for appropriate values of p_{in} and p_{ex}, we have:

(i)* $(1 - o(1))mp_{in}$ edge disjoint Hamilton cycles which contain almost all edges of type 1 in F;

(ii)* $(1 + o(1))mp_{in}$ Hamilton cycles which cover all edges of type 1 in F;

(iii)* $(1 - o(1))^{n-s}(m!)^{\ell-1}p_{in}^{n-s-m}$ directed Hamilton cycles in F.

To do this, we first expose edges of type 1. above. Using known matching results, for $p_{in} = \omega(\log^C m/m)$ and $\ell \leq m$ say, it can be shown that whp for every $j \in [\ell - 1]$:

(i) F_j contains $L_{pack} := (1 - o(1))mp_{in}$ edge disjoint perfect matchings, $\{M_i^j\}_{i=1}^{L_{pack}}$;

(ii) F_j contains $L_{cov} := (1 + o(1))mp_{in}$ perfect matchings covering all edges of F_j, $\{M_i^j\}_{i=1}^{L_{cov}}$;

(iii) F_j contains $(1 - o(1))^m m! p_{in}^m$ perfect matchings.

Now note that in (i), (ii) and (iii) above, by combining a perfect matching from each F_j for each $j \in [\ell - 1]$ we obtain a collection of m vertex disjoint directed paths from V_1 to V_ℓ, covering $\bigcup_{j \in [\ell]} V_j$. We refer to such a collection of paths $\mathcal{P}$ as a *matching path system*.

(i) For each $i \in [L_{pack}]$, by combining the disjoint matchings $\{M_i^j\}_{j=1}^{\ell-1}$ from (i) in this way, we obtain a matching path system $\mathcal{P}_i$. This gives L_{pack} edge disjoint matching path systems $\mathcal{P}_1, \ldots, \mathcal{P}_{L_{pack}}$.

(ii) For each $i \in [L_{cov}]$, by combining the matchings $\{M_i^j\}_{j=1}^{\ell-1}$ from (ii) in this way we obtain a matching path system $\mathcal{P}_i$. This gives L_{cov} matching path systems $\mathcal{P}_1, \ldots, \mathcal{P}_{L_{cov}}$, which cover all edges in the digraphs F_j for $j \in [\ell - 1]$.

(iii) Lastly, by choosing different matching between the partitions from (iii), we have many choices for how to build our matching path system $\mathcal{P}$. We obtain at least $(1 - o(1))^{m\ell}(m!)^{\ell-1}p_{in}^{m(\ell-1)} \geq (1 - o(1))^n(m!)^{\ell-1}p_{in}^{n-s-m}$ such choices for $\mathcal{P}$.

Now let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a fixed matching path system. Assume that each P_i begins at a vertex $s_i \in V_1$ and terminates at a vertex $t_i \in V_\ell$. These vertices are distinct by construction. We will now describe how to include all paths in $\mathcal{P}$ into a directed Hamilton cycle. To do this simply contract each directed path P_i to single vertex which we also denote by P_i. Now expose the edges of type 2. above and view them as edges of a random digraph on vertex set $\widetilde{V} = V_0 \cup \{P_1, \ldots, P_m\}$. Note that the following edges all appear with probability p_{ex}:

- All directed edges in V_0. These come from edges of type 2. (a) above;

- Directed edges from V_0 to $\{P_1, \ldots, P_m\}$ and from $\{P_1, \ldots, P_m\}$ to V_0. These come respectively from edges of type 2. (b) and (c) above;

- Directed edges within the set $\{P_1, \ldots, P_m\}$. These come from edges of type 2 (d) above. (Here we may obtain a loop on the vertices P_i, which we simply ignore.)

6

As all such edges appear independently, the resulting random digraph is distributed identically to $\mathcal{D}(s+m, p_{ex})$. By known Hamiltonicity result for $\mathcal{D}(n, p)$, provided that $p_{ex} = \omega \left(\log^C (m + s)/(m + s) \right)$ we obtain that this digraph is Hamiltonian with very high probability. However, it is easy to see that by construction a directed Hamilton cycle in this contracted digraph pulls back to a directed Hamilton cycle in F, which contains the paths in $\mathcal{P}$ as directed subpaths. Thus we have shown how to turn a single matching path system into a Hamilton cycle.

Now in the case of (ii)*, we can complete each of the matching path systems $\mathcal{P}_1, \ldots \mathcal{P}_{L_{cov}}$ into Hamilton cycles by using edges of type 2. described above. This can also be used to show whp many of the matching paths systems from (iii)* complete to (distinct) directed Hamilton cycles. However, to pack the Hamilton cycles in the case of (i)* more care must be taken as we cannot use the same edges twice. To get around this, we distribute the edges of type 2. to create an individual random digraph for each $\mathcal{P}_i$. Provided that p_{ex} is sufficiently large (and m, ℓ and s are carefully chosen) each of these individual random digraphs will be Hamiltonian whp. This completes the description of (i)*, (ii)* and (iii)* above.

Now our approach for dealing with the packing, covering and counting problems on $\mathcal{D}(n, p)$ is to show that with high probability we can break $D \sim \mathcal{D}(n, p)$ into subdigraphs distributed similarly to F above. However the type of decomposition chosen is again dependent on the problem at hand. With the packing it is important that these graphs are edge disjoint. With the covering, it will be important every edge of $\mathcal{D}(n, p)$ appears as an edge of type 1. in one of these digraphs (recall these were the only edges guaranteed to be covered in (ii)*). The counting argument is less sensitive, and simply work with many such digraphs. Dependent on the problem, we can apply our strategy above for $F \sim \mathcal{F}$ to each of these digraphs separately. Combining the resulting Hamilton cycles from either (i)*, (ii)* or (iii)* in each of these digraphs will then solve the corresponding problem for $\mathcal{D}(n, p)$.

2.2 Probabilistic tools

We will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the binomial distribution due to Chernoff (see [1], [22]).

Lemma 2.1 (Chernoff's inequality). *Let $X \sim \mathrm{Bin}(n, p)$ and let $\mu = \mathbb{E}(X)$. Then*

- $\Pr[X < (1 - a)\mu] < e^{-a^2 \mu/2}$ *for every $a > 0$;*

- $\Pr[X > (1 + a)\mu] < e^{-a^2 \mu/3}$ *for every $0 < a < 3/2$.*

Remark 2.2. *The conclusions of Chernoff's inequality remain the same when X has the hypergeometric distribution (see [22], Theorem 2.10).*

We will also find the following bound useful.

Lemma 2.3. *Let $X \sim Bin(n, p)$. Then $\Pr[X \geq k] \leq \left(\frac{enp}{k} \right)^k$.*

Proof. Just note that

$$\Pr[X \geq k] \leq \binom{n}{k} p^k \leq \left(\frac{enp}{k} \right)^k.$$

□

2.3 Perfect matchings in bipartite graphs and random bipartite graphs

The following lower bound on the number of perfect matchings in an r-regular bipartite graph is also known as the Van der Waerden conjecture and has been proven by Egorychev [9] and by Falikman [10]:

Theorem 2.4. *Let $G = (A \cup B, E)$ be an r-regular bipartite graph with parts of sizes $|A| = |B| = n$. Then, the number of perfect matchings in G is at least $\left(\frac{r}{n}\right)^n n!$.*

The following lemma is an easy corollary of the so called Gale-Ryser theorem (see, e.g. [29]).

Lemma 2.5. *(Lemma 2.4, [12]) Let G is a random bipartite graph between two vertex sets both of size n, where edges are chosen independently with probability $p = \omega(\log n/n)$. Then with probability $1 - o(1/n)$ the graph G contains $(1 - o(1))np$ edge disjoint perfect matchings.*

2.4 Converting paths into Hamilton cycles

The following definitions will be convenient in our proofs.

Definition 2.6. *Suppose that X is a set of size n and that ℓ, m, s are positive integers with $n = m\ell + s$. A sequence $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ of subsets of X is called an (ℓ, s)-partition of X if*

- *$X = V_0 \cup V_1 \cup \ldots \cup V_\ell$ is a partition of X, and*

- *$|V_0| = s$, and*

- *$|V_i| = m$ for every $i \in [\ell]$.*

Definition 2.7. *Given an (ℓ, s)-partition $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ of a set X, let $D_n(\mathcal{V})$ denote the digraph on vertex set $X = [n]$ consisting of all edges $\overrightarrow{uv}$ such that:*

1. *$u \in V_j$, $v \in V_{j+1}$ for some $j \in [\ell - 1]$, or*

2. *$u \in V_0$ and $v \in V_0 \cup V_1$, or*

3. *$u \in V_\ell$, $v \in V_0 \cup V_1$.*

We call edges of type 1. interior edges *and call edges of type 2. and 3.* exterior edges.

Suppose that we are given two disjoint sets V and W and a digraph D on vertex set $V \cup W$. Suppose also that we have m disjoint ordered pairs $\mathcal{M} = \{(w_i, x_i) : i \in [m]\} \subset W \times W$. Then we define the following auxiliary graph.

Definition 2.8. *Let $D(\mathcal{M}, V)$ denote the following auxiliary digraph on vertex set $\mathcal{M} \cup V$ where $\mathcal{M} = \{u_1, \ldots, u_m\}$ and each u_i refers to the pair (w_i, x_i). Then given any two vertices $v_1, v_2 \in V$, we have:*

- *$\overrightarrow{v_1 v_2}$ is an edge in $D(\mathcal{M}, V)$ if it appears in D;*

- *$\overrightarrow{v_1 u_i}$ is an edge in $D(\mathcal{M}, V)$ if $\overrightarrow{v_1 w_i}$ is an edge in D;*

- *$\overrightarrow{u_i v_1}$ is an edge in $D(\mathcal{M}, V)$ if $\overrightarrow{x_i v_1}$ is an edge in D;*

- *$\overrightarrow{u_i u_j}$ is an edge in $D(\mathcal{M}, V)$ if $\overrightarrow{x_i w_j}$ is an edge in D.*

Remark 2.9. *Note that if $D(\mathcal{M}, V)$ contains a directed Hamilton cycle and W can be decomposed into vertex disjoint directed $w_i x_i$-paths for all $i \in [m]$ (paths starting at w_i and ending at x_i) then D contains a directed Hamilton cycle.*

3 Counting Hamilton cycles in $\mathcal{D}(n,p)$

In this section we prove Theorem 1.3. The proof of this theorem is relatively simple and contains most of the ideas for the other main results and therefore serves as a nice warmup.

Proof. We will first prove the upper bound. For this, let X_H denote the random variable that counts the number of Hamilton cycles in $D \sim \mathcal{D}(n,p)$. It is clear that $\mathbb{E}[X_H] = (n-1)! p^n$. By Markov's inequality, we therefore have

$$Pr(X_H \geq (1+o(1))^n n! p^n) \leq \frac{\mathbb{E}[X_H]}{(1+o(1))^n n! p^n} = (1-o(1))^n = o(1).$$

Thus $X_H \leq (1+o(1))^n n! p^n$ w.h.p..

We now prove the lower bound, i.e. $X_H \geq (1-o(1))^n n! p^n$ w.h.p.. Let $\alpha := \alpha(n)$ be a function tending to infinity arbitrarily slowly with n. We prove the lower bound on X_H under the assumption that $p \geq \alpha^2 \log^2 n / n$. Let us take s and ℓ to be integers where s is roughly $\frac{n}{\alpha \log n}$ and ℓ is roughly $2\alpha \log n$ and there is an integer m with $n = \ell m + s$. Also fix a set $S \subseteq V(G)$ of order s and let us set $V' = V(D) \setminus S$. The set S will be used to turn collections of vertex disjoint paths into Hamilton cycles.

To begin, take a fixed (ℓ, s)-partition $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ with $V_0 = S$. We claim the following:

Claim: Given $\mathcal{V}$ as above, taking $D \sim \mathcal{D}(n,p)$, the random digraph $D \cap D_n(\mathcal{V})$ (where $D_n(\mathcal{V})$ is as in Definition 2.7) contains at least $(1-o(1))^n m!^{\ell-1} p^{m(\ell-1)}$ distinct Hamilton cycles with probability $1 - o(1)$.

To see this, first expose the interior edges of $D \cap D_n(\mathcal{V})$. For each $j \in [\ell-1]$ let $F_j := E_D(V_j, V_{j+1})$. Observe that $F_j \sim D(m, m, p)$. It will be convenient for us to view F_j as a bipartite graph obtained by ignoring the edge directions. Since $p = \omega\left(\frac{\log n}{m}\right)$, by Lemma 2.5 with probability $1 - o(1/n)$ we conclude that F_j contains $(1-o(1))mp$ edge-disjoint perfect matchings. Taking a union bound over all $j \in [\ell-1]$ we find that whp F_j contains a $(1-o(1))mp$-regular subgraph for all $j \in [\ell-1]$.

Apply Theorem 2.4 to each of these subgraphs. This give that for each $j \in [\ell-1]$ the graph F_j contains at least $(1-o(1))^m m! p^m$ perfect matchings. Combining a perfect matching from each of the F_j's we obtain a family $\mathcal{P}$ of m vertex disjoint paths which spans V'. Let $\Lambda_\mathcal{V}$ denote the set of all such $\mathcal{P}$. From the choices of perfect matchings in each F_j we obtain that whp

$$|\Lambda_\mathcal{V}| \geq ((1-o(1))^m m! p^m)^{\ell-1} = (1-o(1))^n (m!)^{\ell-1} p^{n-s}. \tag{1}$$

Now let $\mathcal{P} = \{P_1, \ldots, P_m\} \in \Lambda_\mathcal{V}$. Let

$$\mathcal{M} = \{(u_i, v_i) \in V_1 \times V_\ell : P_i \text{ is a } u_i - v_i \text{ directed path}\}.$$

Let us consider the auxiliary digraph $D(\mathcal{M}, V_0)$ as in Definition 2.8. As we expose the exterior edges of D in $D_n(\mathcal{V})$ it is easy to see that $D(\mathcal{M}, V_0) \sim \mathcal{D}(s+m, p)$. Furthermore, a Hamilton cycle in $D(\mathcal{M}, V_0)$ gives a Hamilton cycle in D by Remark 2.9. However, it is well-known digraphs in $\mathcal{D}(n, p')$ are Hamiltonian w.h.p. for say $p' > 2 \log n / n$ ([16]). Since

$$p = \frac{\alpha^2 \log^2 n}{n} \geq \frac{\alpha \log n}{s+m} = \omega\left(\frac{\log(s+m)}{s+m}\right)$$

9

we find that $D(\mathcal{M}, V_0)$ is Hamiltonian w.h.p.. Thus we have shown that

$$Pr\Big(\mathcal{P} \text{ does not extend to a Hamilton cycle in } D \cap D_n(\mathcal{V})\Big) = o(1). \tag{2}$$

Let $\Lambda'_\mathcal{V} \subset \Lambda_\mathcal{V}$ denote the set of $\mathcal{P} \in \Lambda_\mathcal{V}$ which do not extend to a Hamilton cycle in $D \cap D_n(\mathcal{V})$. By (2) we have $\mathbb{E}(|\Lambda'_\mathcal{V}|) = o(|\Lambda_\mathcal{V}|)$. Using Markov's inequality we obtain that $|\Lambda'_\mathcal{V}| = o(|\Lambda_\mathcal{V}|)$ whp. Combined with (1) this gives that $|\Lambda_\mathcal{V} \setminus \Lambda'_\mathcal{V}| \geq (1 - o(1))^n \, (m!)^{\ell-1} p^{n-s}$ whp. Lastly, to complete the proof of the claim, note that any two distinct families $\mathcal{P}, \mathcal{P}' \in \Lambda_\mathcal{V} \setminus \Lambda'_\mathcal{V}$ yield different Hamilton cycles – indeed, by deleting the vertices of S from the Hamilton cycle it is easy to recover the paths $\mathcal{P}$. This proves the claim.

Now to complete the proof of the theorem, let Γ denote the set of (ℓ, s)-partitions $\mathcal{V}$ with $V_0 = S$ which satisfy the statement of the claim. By Markov's inequality we have $|\Gamma| \geq (1 - o(1))\frac{(n-s)!}{(m!)^\ell}$ whp. Since for distinct $\mathcal{V}, \mathcal{V}' \in \Gamma$ the Hamilton cycles in $D \cap D_n(\mathcal{V})$ are all distinct, we find that whp D contains at least

$$|\Gamma|(1 - o(1))^n \, (m!)^{\ell-1} p^{n-s} \geq (1 - o(1))^n \frac{(n-s)!}{(m!)^\ell}(m!)^{\ell-1}p^n$$

$$= (1 - o(1))^n \frac{(n-s)!}{m!}p^n = (1 - o(1))^n n! p^n$$

distinct Hamilton cycles. The final equality here holds since $m < n/\alpha \log n$ gives that $m! < e^{n/\alpha}$ and $(n)_s \leq n^s = (1 + o(1))^n$ since $s = o(n/\log n)$. This completes the proof of the theorem. $\qquad \square$

4 Packing Hamilton cycles in $\mathcal{D}(n, p)$

In this section we prove Theorem 1.1. The heart of the argument is contained in the following lemma.

Lemma 4.1. *Let* $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ *be an* (ℓ, s)-*partition of a set* X *of size* $n = \ell m + s$. *Suppose that we select a random subdigraph* F *of* $D_n(\mathcal{V})$ *as follows:*

- *include each interior directed edge of* $D_n(\mathcal{V})$ *independently with probability* p_{in};

- *include each exterior directed edge of* $D_n(\mathcal{V})$ *independently with probability* p_{ex}.

Then, provided $p_{in} = \omega(\log n/m)$ *and* $p_{ex} = \omega(mp_{in} \log n/(m+s))$, *w.h.p.* F *contains* $(1 - o(1))mp_{in}$ *edge-disjoint Hamilton cycles.*

Proof. We begin by exposing the interior edges of F. For $j \in [\ell - 1]$ all edges $E_{D_n}(V_j, V_{j+1})$ appear in $E_F(V_j, V_{j+1})$ independently with probability p_{in}. By ignoring the orientations, we can view $E_F(V_j, V_{j+1})$ as a bipartite graph. From Lemma 2.5, since $p_{in} = \omega(\log m/m)$ we find that w.h.p. for all $j \in [\ell - 1]$ the graph $E_F(V_j, V_{j+1})$ contains $L := (1 - o(1))mp_{in}$ edge-disjoint perfect matchings $\{\mathcal{M}_{j,k}\}_{k=1}^L$. For each $k \in [L]$, combining the edges in the matchings $\{\mathcal{M}_{j,k}\}_{j\in[\ell-1]}$ gives m directed paths, each directed from V_1 to V_ℓ and covering $\bigcup_{i=1}^\ell V_i$. Let $P_{k,1}, \ldots, P_{k,m}$ denote these paths and $\mathcal{P}_k = \{P_{k,1}, \ldots, P_{k,m}\}$.

Now for each exterior edge e of $D_n(\mathcal{V})$ choose a value $h(e) \in [L]$ uniformly at random, all values chosen independently. Now expose the exterior edges of F and for each $i \in [L]$ let H_i denote the subgraph of F with edge set $\{e \in E(F) : e \text{ exterior with } h(e) = i\}$.

Claim 4.2. *For any $k \in [L]$, the digraph $C_k := \{ \vec{e} : \vec{e}$ is a directed edge of some path in $\mathcal{P}_k \} \cup H_k$ contains a directed Hamilton cycle with probability $1 - o(1/n)$.*

Note that the proof of the lemma follows from the claim, by taking a union bound over all $k \in [L]$.

To prove the claim, let

$$\mathcal{M}_k = \{(u_{k,i}, v_{k,i}) \in V_1 \times V_\ell : P_{k,i} \text{ is a directed } u_{k,i} - v_{k,i} \text{ path}\}.$$

By Remark 2.9 it suffices to prove that the auxiliary digraph $C_k(\mathcal{M}_k, V_0)$ contains a directed Hamilton cycle. Note that $|V(C_k(\mathcal{M}_k, V_0))| = s + m$ and that $C_k(\mathcal{M}_k, V_0) \sim \mathcal{D}(s + m, p_{ex}/L)$. Since $p_{ex}/L = \omega(\log n/(s+m))$, the digraph $C_k(\mathcal{M}_k, V_0)$ is Hamiltonian with probability $1 - o(1/n)$. By Remark 2.9, this completes the proof of the claim, and therefore the lemma. $\square$

The following lemma allows us to cover the edges of the complete digraph in a reasonably balanced way using copies of $D_n(\mathcal{V})$.

Lemma 4.3. *Suppose that X is a set of size n and that $\ell, m, s \in \mathbb{N}$ satisfying $n = m\ell + s$, with $t = \omega(\ell \log n)$, $t = \omega((n^2/s^2) \log n)$ and $s = o(n)$ and $m = o(s)$. Let $\mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(t)}$ be a collection of (ℓ, s)-partitions of X chosen uniformly and independently at random, where $\mathcal{V}^{(i)} = (V_0^{(i)}, \ldots, V_\ell^{(i)})$. Then w.h.p. for each pair $u, v \in X$, the directed edge $e = \vec{uv}$ satisfies:*

1. *$|A_e| = (1 + o(1))\frac{t}{\ell}$ where $A_e := \{i \in [t] : e$ is an interior edge of $D_n(\mathcal{V}^{(i)})\}$.*

2. *$|B_e| = (1 + o(1))\frac{s^2 t}{n^2}$ where $B_e := \{i \in [t] : e$ is an exterior edge of $D_n(\mathcal{V}^{(i)})\}$.*

Proof. Let $\mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(t)}$ be (ℓ, s)-partitions chosen uniformly and independently at random. Given a fixed directed edge e, the sizes $|A_e|$ and $|B_e|$ are binomially distributed on a set t with

$$\mathbb{E}(|A_e|) = (\ell - 1)\frac{m^2}{n(n-1)}t \quad \text{and} \quad \mathbb{E}(|B_e|) = \frac{m^2 + 2sm + s(s-1)}{n(n-1)}t.$$

Using that $s = o(n)$ and $n = m\ell + s$ this gives that $\mathbb{E}(|A_e|) = (1 + o(1))\frac{t}{\ell}$ and using $m = o(s)$ gives $\mathbb{E}(|B_e|) = (1 + o(1))\frac{s^2 t}{n^2}$. Therefore by Lemma 2.1

$$\Pr\left(\big||A_e| - \mathbb{E}(|A_e|)\big| > a\mathbb{E}(|A_e|)\right) \leq 2e^{-a^2\mathbb{E}(|A_e|)/3} \leq 2e^{-(1+o(1))a^2 t/3\ell} = o(1/n^2). \tag{3}$$

Here we used that $a^2 t/3\ell \geq 3 \log n$ for $a = o(1)$. Similarly using that $ts^2/n^2 = \omega(\log n)$ we find $\Pr\left(\big||B_e| - \mathbb{E}(|B_e|)\big| > a\mathbb{E}(|B_e|)\right) = o(1/n^2)$. Taking a union bound over all directed edges gives that w.h.p. 1. and 2. hold for all e. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\alpha = \alpha(n)$ be some function tending to infinity arbitrarily slowly with n. Suppose that $p \geq \alpha^6 \log^4 n/n$ and let $\ell = \alpha^3 \log n$, $s = n/\alpha^2 \log n$ be integers with $n = m\ell + s$. Note that this gives $m = (1 + o(1))n/\alpha^3 \log n$. Additionally set $t = \alpha^5 \log^3 n$. With these choices, the hypothesis of Lemma 4.3 is satisfied. Let $\mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(t)}$ be a collection of (ℓ, s)-partitions of $X = [n]$, chosen so that the conclusions of Lemma 4.3 are satisfied. Therefore $|A_e| = (1 + o(1))t/\ell = (1 + o(1))\alpha^2 \log^2 n$ and $|B_e| = (1 + o(1))s^2 t/n^2 = (1 + o(1))\alpha \log n$ for every e.

11

To begin, whenever we expose the edges of a directed graph $D \sim \mathcal{D}(n,p)$, we will assign the edges of D among t edge disjoint subdigraphs $D^{(1)}, \ldots, D^{(t)}$. The digraphs $D^{(i)}$ are constructed as follows. For each edge e independently choose a random value $h(e) \in A_e \cup B_e$ where an element in A_e is selected with probability $(1 - 1/\alpha)/|A_e|$ and an element in B_e is selected with probability $1/\alpha|B_e|$. For each $i \in [t]$, we take $D^{(i)}$ to be the digraph given by $D^{(i)} = \{e \in E(D) : h(e) = i\}$. We prove that w.h.p. $D^{(i)}$ contains edge-disjoint Hamilton cycles covering almost all of its edges.

First note that all edges e of $D_n(\mathcal{V}^{(i)})$ appear independently in $D^{(i)}$. If e is an interior edge then the probability that it appears is $p(1 - 1/\alpha)/|A_e| \geq (1 - o(1))p/\alpha^2 \log^2 n := p_{in}$. Similarly, each exterior edge e in $D_n(\mathcal{V}^{(i)})$ appears in $D^{(i)}$ with probability $p/\alpha|B_e| \geq (1 - o(1))p/\alpha^2 \log n := p_{ex}$. Using these values, select F as in Lemma 4.1. Also set $L = (1 - o(1))mp_{in}$. Due to monotonicity we conclude that for every $i \in [t]$ we have

$$\Pr(D^{(i)} \text{ contains } L \text{ edge disjoint Ham. cycles}) \geq \Pr(F \text{ contains } L \text{ edge disjoint Ham. cycles}). \quad (4)$$

Now we claim with these choices of p_{in} and p_{ex} the hypothesis of Lemma 4.1 are satisfied. Indeed, using $p \geq \alpha^6 \log^4 n/n$ gives

$$(1 + o(1))p_{in} = \frac{p}{\alpha^2 \log^2 n} \geq \frac{\alpha^4 \log^2 n}{n} = (1 + o(1))\frac{\alpha \log n}{m},$$

and so $p_{in} = \omega(\log n/m)$. Similarly we have

$$p_{ex} = (1 + o(1))\frac{p}{\alpha^2 \log n} = (1 + o(1))p_{in} \log n = (1 + o(1))\frac{\alpha m p_{in} \log n}{s},$$

and $p_{ex} = \omega(mp_{in} \log n/(m+s))$. Thus by Lemma 4.1, $\Pr(F \text{ contains } L \text{ edge disjoint Ham. cycles}) = 1 - o(1)$. Summing over $i \in [t]$ and combining with (4), this proves that w.h.p. D contains at least $(1 - o(1))Lt = (1 - o(1))mp_{in}t = (1 - o(1)).\frac{n-s}{\ell}.\frac{p\ell}{t}.t = (1 - o(1))np$ edge-disjoint Hamilton cycles. $\square$

5 Covering $\mathcal{D}(n,p)$ with Hamilton cycles

In this section we prove Theorem 1.2. To begin we first prove the following lemma. The proof makes use of the max-flow min-cut theorem and the integrality theorem for network flows (see Chapter 7 in [37]).

Lemma 5.1. *Let $G = (A, B, E)$ be a bipartite graph, with $|A| = |B| = N$ and $\delta(G) \geq d$. Suppose that G has the following properties:*

- *For any $X \subset A$, $Y \subset B$ with $|X| \geq \frac{N}{4}$ and $|Y| \geq \frac{N}{4}$ we have $e_G(X,Y) \geq \frac{dN}{40}$,*

- *For any $X \subset A$ with $|X| \leq \frac{N}{4}$, if $e_G(X,Y) \geq \frac{3d|X|}{4}$ for some $Y \subset B$ then $|Y| \geq 2|X|$,*

- *For any $Y \subset B$ with $|Y| \leq \frac{N}{4}$, if $e_G(X,Y) \geq \frac{3d|Y|}{4}$ for some $X \subset A$ then $|X| \geq 2|Y|$.*

Then given any integer r with $r \leq \frac{d}{80}$ and a bipartite graph H on vertex set $A \cup B$ with $\Delta := \Delta(H) \leq \frac{r}{2}$, there exists a subgraph G' of G which is edge disjoint from H such that $G' \cup H$ is r-regular.

Proof. Given a graph F on $V(G)$ and a vertex v of G, let $d_F(v)$ denote the degree of v in F. By assumption, we have $d_H(v) \leq r/2$ for all $v \in A \cup B$. We wish to find a subgraph G' of G which is edge-disjoint from H so that $d_{G'}(v) + d_H(v) = r$ for all $v \in V(G)$. We prove the existence of G' by representing it as a flow in an appropriate network.

Consider the following network D on vertex set $V(G) \cup \{s, t\}$, with source s and sink t. For each $a \in A$, the edge $\overrightarrow{sa} \in E(D)$ and it has capacity $r - d_H(a)$. For each $b \in B$, the edge $\overrightarrow{bt} \in E(D)$ and it has capacity $r - d_H(b)$. Lastly, each edge in $E(G \setminus H)$ is directed from A to B and has capacity 1. Using the integrality theorem for network flows, it is sufficient to show that there is a flow from s to t of value

$$V = \sum_{a \in A} (r - d_H(a)) = rN - \sum_{a \in A} d_H(a). \tag{5}$$

By the max-flow min-cut theorem it is sufficient to show that D does not contain an $s - t$ cut of capacity less than V.

To see this, suppose for contradiction that $\{s\} \cup A_s \cup B_s$ and $A_t \cup B_t \cup \{t\}$ forms such a cut, $A_v \subset A$ and $B_v \subset B$ for $v \in \{s, t\}$. The capacity of this cut is

$$C = \sum_{a \in A_t} (r - d_H(a)) + \sum_{b \in B_s} (r - d_H(b)) + e_{G \setminus H}(A_s, B_t).$$

We may assume that $|A_s| \leq N/4$ or $|B_t| \leq N/4$. Indeed, otherwise from the statement of the lemma we have $e_G(A_s, B_t) \geq dN/40$ and

$$C \geq e_{G \setminus H}(A_s, B_t) \geq e_G(A_s, B_t) - \Delta N \geq dN/40 - rN/2 \geq rN \geq V,$$

since $r \leq d/80$. We will focus on the case $|A_s| \leq N/4$ as the case $|B_t| \leq N/4$ follows from an identical argument.

Note that since $e_{G \setminus H}(A_s, B) \geq (\delta(G) - \Delta)|A_s| \geq (d - \Delta)|A_s|$, we find

$$e_{G \setminus H}(A_s, B_t) \geq e_{G \setminus H}(A_s, B) - e_{G \setminus H}(A_s, B_s) \geq (d - \Delta)|A_s| - e_G(A_s, B_s). \tag{6}$$

From (6) it follows that if $e_G(A_s, B_s) \leq \frac{3d|A_s|}{4}$ then

$$C \geq \sum_{a \in A_t} (r - d_H(a)) + e_{G \setminus H}(A_s, B_t) \geq \sum_{a \in A_t} (r - d_H(a)) + (d - \Delta - \frac{3d}{4})|A_s|$$

$$\geq \sum_{a \in A_t} (r - d_H(a)) + r|A_s| \geq V,$$

where the second last inequality holds since $d/4 \geq 2r \geq \Delta + r$ and the last inequality holds by (5). If $e_G(A_s, B_s) \geq \frac{3d|A_s|}{4}$, since $|A_s| \leq |A|/4$, by the hypothesis of the lemma we have $|B_s| \geq 2|A_s|$. But then, since $\Delta \leq r/2$ we have

$$C \geq \sum_{a \in A_t} (r - d_H(a)) + \sum_{b \in B_s} (r - d_H(b)) \geq \sum_{a \in A_t} (r - d_H(a)) + |B_s|(r - \Delta)$$

$$\geq \sum_{a \in A_t} (r - d_H(a)) + 2|A_s| \times \frac{r}{2}$$

$$\geq \sum_{a \in A} (r - d_H(a)) = V.$$

This covers all cases, and completes the proof. $\square$

13

We now prove a covering version of Lemma 4.1. In our proof of Theorem 1.2 we will again break $D \sim \mathcal{D}(n,p)$ into many sub-digraphs which are distributed similarly to F from Lemma 4.1. However there will be some small fluctuation in the edge probabilities of edges in these sub-digraphs. The slightly unusual phrasing of the next lemma is intended to allow for these fluctuations.

Lemma 5.2. *Let $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ be an (ℓ, s)-partition of a set X of size $n = m\ell + s$. Choose a random subdigraph F of $D_n(\mathcal{V})$ as follows:*

- *include each interior edge e from $D_n(\mathcal{V})$ independently with probability $q_e \in (1 \pm o(1))p_{in}$;*

- *include each exterior edge from $D_n(\mathcal{V})$ independently with probability at least p_{ex}.*

Then, provided $p_{in} = \omega(\log n/m)$ and $p_{ex} = \omega(\log n/(m+s))$, with probability $1 - o(1/n^2)$ there are $(1 + o(1))mp_{in}$ directed Hamilton cycles in F which cover all interior edges of $F \cap D_n(\mathcal{V})$.

Proof. We begin by exposing the interior edges of F. For any $j \in [\ell - 1]$, all of edges $E_{D_n}(V_j, V_{j+1})$ appear in $E_F(V_j, V_{j+1})$ independently with probability at least $(1 - o(1))p_{in}$. For any $j \in [\ell - 1]$, let F_j be the subdigraph of F consists of the vertices $V_j \cup V_{j+1}$ and the edges in $E_F(V_j, V_{j+1})$). We again view F_j as a bipartite graph, simply by ignoring the orientations. As in Lemma 4.1, with probability $1 - o(1/n^2)$ for each $j \in [\ell - 1]$ we can find $L = (1 - o(1))mp_{in}$ edge-disjoint perfect matchings in $E_F(V_j, V_{j+1})$, which we denote by $\{\mathcal{M}_{j,k}\}_{k=1}^{L}$. Now remove the edges of these matchings from $E_F(V_j, V_{j+1})$ and let H_j denote the remaining subdigraph. Since $p_{in} = \omega(\log n/m)$ and $q \in (1 + o(1))p_{in}$, by Chernoff's inequality, with probability $1 - o(1/n^2)$ every vertex $u \in V_j$ and $v \in V_{j+1}$ satisfies

$$(1 + o(1))mp_{in} \le d_{F_j}^+(u), d_{F_j}^-(v) \le (1 + o(1))mp_{in}.$$

Therefore with probability $1 - o(1/n^2)$, for all $j \in [\ell - 1]$, such u and v satisfy

$$d_{H_j}^+(u) = o(mp_{in}) \quad \text{and} \quad d_{H_j}^-(v) = o(mp_{in}). \tag{7}$$

Now given $X \subset V_j$ and $Y \subset V_{j+1}$ we also have $\mathbb{E}(e_{F_j}(X, Y)) = (1 \pm o(1))|X||Y|p_{in}$. Chernoff's inequality therefore shows that

$$Pr\left(\left|e_{F_j}(X, Y) - (1 \pm o(1))|X||Y|p_{in}\right| > t\right) \le e^{-t^2/4|X||Y|p_{in}}.$$

Using this bound it is easy to check that the following holds: with probability $1 - n^{\omega(1)}$, for all $j \in [\ell - 1]$ the hypothesis of Lemma 5.1 are satisfied by the bipartite graph F_j, taking $d = (1 - o(1))mp_{in}$ and $N = m$. Setting $r = \max_{j \in [\ell - 1]}\{2\Delta(H_j)\}$, from (7) we have $r \ll d$ for all $j \in [\ell - 1]$. Therefore by Lemma 5.1, with probability $1 - n^{-\omega(1)}$, for all $j \in [\ell - 1]$ the graph F_j contains an r-regular subgraph G_j which includes all edges of H_j.

Now by Hall's theorem, for each $j \in [\ell - 1]$ the digraph G_j can be decomposed into r edge-disjoint perfect matchings, which we denote by $\{M_{j,k}\}_{k=L+1}^{L+r}$. Combined with the matchings at the beginning of the proof, we have show that with probability $1 - o(1/n^2)$, for each $j \in [\ell - 1]$ there are perfect matchings $\{\mathcal{M}_{j,k}\}_{k=1}^{L+r}$ which cover all interior directed edges of F. By combining the edges $\{\mathcal{M}_{j,k}\}_{j=1}^{\ell-1}$ for each $k \in [L + r]$, we get m directed paths, each directed from V_1 to V_ℓ and covering $\bigcup_{i=1}^{\ell} V_i$. Let $P_{k,1}, \ldots, P_{k,m}$ denote these paths and $\mathcal{P}_k = \{P_{k,1}, \ldots, P_{k,m}\}$. In particular these paths cover all interior edges of F.

14

Now to complete the proof we expose the exterior edges of F and use them to complete each $\mathcal{P}_k$ into a directed Hamilton cycle as in the proof of Lemma 4.1. For each $k \in [L+r]$ let

$$\mathcal{M}_k = \{(u_{k,i}, v_{k,i}) \in V_1 \times V_\ell : P_{k,i} \text{ is a directed } u_{k,i} - v_{k,i} \text{ path}\}.$$

Now $|V(F(\mathcal{M}_k, V_0))| = s + m$ and as in Lemma 4.1, $F(\mathcal{M}_k, V_0) \sim \mathcal{D}(s + m, p_{ex})$. Since $p_{ex} = \omega(\log n/(s+m))$, the digraph $F(\mathcal{M}_k, V_0)$ is Hamiltonian with probability $1 - o(1/n^3)$. By Remark 2.9, this shows that with probability $1 - o(1/n^2)$, for all $k \in [L+r]$ the digraph F contains a directed Hamilton cycle containing all edges of the paths in $\mathcal{P}_k$. As these paths cover all interior edges of F, this completes the proof of the lemma. $\square$

Proof of Theorem 1.2. The proof follows a similar argument to that of Theorem 1.1. Let $\alpha = \alpha(n)$ be some function tending arbitrarily slowly to infinity with n and let $p \geq \alpha^4 \log^2 n/n$. Let $n = m\ell + s$ where $\ell = \alpha$, $s = n/\alpha$ and $t = \alpha^2 \log n$. Note that $m = (1 + o(1))n/\alpha$.

Since $t = \omega(\ell \log n)$ we can take $\mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(t)}$ to be a collection of (ℓ, s)-partitions of $X = [n]$ as given by Lemma 4.3. To begin, whenever we expose the edges of a directed graph $D \sim \mathcal{D}(n, p)$, we will assign the edges among t sub-digraphs $D^{(1)}, \ldots, D^{(t)}$. The digraphs $D^{(i)}$ are constructed as follows. Let

$$A_e := \{i \in [t] : e \text{ is interior in } D_n(\mathcal{V}^{(i)})\}.$$

By Lemma 4.3, w.h.p. for each edge e we have $|A_e| = (1+o(1))t/\ell = (1+o(1)\alpha \log n$. Independently for each edge e choose a value $h(e) \in A_e$ uniformly at random. For each $i \in [t]$, let the digraph $D^{(i)}$ contain the edges $\{e \in E(D) : h(e) = i\}$. Furthermore, adjoin all edges of D which occur as an exterior edge of $D_n(\mathcal{V}^{(i)})$ to $D^{(i)}$. We will prove that w.h.p. $D^{(i)}$ contains directed Hamilton cycles covering all the edges of $D \cap D_n(\mathcal{V}^{(i)})$.

First note that all edges e of $D_n(\mathcal{V}^{(i)})$ appear independently in $D^{(i)}$. If e is an interior edge then the probability that it appears is $p/|A_e| = (1 \pm o(1))p/\alpha \log n = p_{in}$. We see that each interior edge of $D_n(\mathcal{V}^{(i)})$ appears in $D^{(i)}$ independently with probability between $(1 - o(1))p_{in}$ and $(1 + o(1))p_{in}$. Also, each exterior edge e in $D_n(\mathcal{V}^{(i)})$ appears in $D^{(i)}$ with probability $p_{ex} := p$. Now we have $p_{ex} = p = \omega(\frac{\log n}{m+s})$. We also have $p_{in} = (1 + o(1))\frac{p\ell}{t} \geq \frac{\alpha^2 \log n}{n} = \frac{\alpha \log n}{m}$, so $p_{in} = \omega(\frac{\log n}{m})$. Thus by Lemma 5.2 we obtain

$$\Pr(D^{(i)} \text{ has } (1 + o(1))mp_{in} \text{ directed Hamilton cycles covering its interior edges}) \geq 1 - \frac{1}{n^2}. \quad (8)$$

Summing (8) over $i \in [t]$, this proves that w.h.p. D contains $(1+o(1))mp_{in}t = (1+o(1))np$ Hamilton cycles covering the interior edges of $D^{(i)}$ for all $i \in [t]$. Since each edge of D occurs as an interior edge of $D^{(i)}$ for some $i \in [t]$, this completes the proof of the theorem. $\square$

6 Packing Hamilton cycles in pseudo-random directed graphs

6.1 Pseudo-random digraphs and Hamiltonicity

Definition 6.1. *A directed graph D on n vertices is called (n, λ, p)-pseudo-random if the following hold:*

(P1) $(1 - \lambda)np \leq d_D^+(v), d_D^-(v) \leq (1 + \lambda)np$ *for every $v \in V(D)$;*

(P2) *For every $X \subseteq V(D)$ of size $|X| \leq \frac{4\log^8 n}{p}$ we have $e_D(X) \leq (1 - \lambda)|X|\log^{8.02} n$;*

(P3) *For every two disjoint subsets $X, Y \subseteq V(D)$ of sizes $|X|, |Y| \geq \frac{\log^{1.1} n}{p}$ we have $e_D(X, Y) = (1 \pm \lambda)|X||Y|p$.*

The following theorem of Ferber, Nenadov, Noever, Peter and Skorić [13] gives a sufficient condition for pseudo-random digraph to be Hamiltonian.

Theorem 6.2. *(Theorem 3.2, [13]) Let $0 < \lambda < 1/10$. Then for $p = \omega(\frac{\log^8 n}{n})$ the following holds. Let D be a directed graph with the following properties:*

(P1) $(1 - \lambda)np \leq d_D^+(v), d_D^-(v) \leq (1 + \lambda)np$ *for every $v \in V(D)$;*

(P2)* *for every $X \subseteq V(D)$ of size $|X| \leq \frac{\log^2 n}{p}$ we have $e_D(X) \leq |X| \log^{2.1} n$;*

(P3)* *for every two disjoint subsets $X, Y \subseteq V(D)$ of sizes $|X|, |Y| \geq \frac{\log^{1.1} n}{p}$ we have $e_D(X, Y) \leq (1 + \lambda)|X||Y|p$.*

Then D contains a Hamilton cycle.

6.2 Properties of pseudo-random graphs

The following lemmas will be useful in the proof of Theorem 1.4. We have deferred the proofs to the Appendix. In these lemmas we assume that $p = \omega(\log^{14} n/n)$, $p' = p/\log^6 n$, $s = \sqrt{n/\alpha p'}$ and $m = s/\log n$.

Lemma 6.3. *Let D be a (n, λ, p)-pseudo-random digraph with $0 < \lambda < 1$ and $p = \omega(\log^{14} n/n)$. We first select a random subdigraph C of D by including edges independently with probability $q \in (1 \pm o(1))p'/p$. Then select an (ℓ, s)-partition of $V(D)$ given by $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ uniformly at random, with $|V_0| = s$ and $n = m\ell + s$. Then with probability $1 - o(1/n)$ the following holds: for every collection $\mathcal{M}$ of m disjoint pairs from $V_1 \times V_\ell$, the random digraph $F_0 = C(\mathcal{M}, V_0)$ satisfies the following properties:*

(A) $(1 - 3\lambda)(s + m)p' \leq d_{F_0}^+(v), d_{F_0}^-(v) \leq (1 + 3\lambda)(s + m)p'$ for every $v \in V(F_0)$,

(B) we have $e_{F_0}(X) \leq |X| \log^{2.1} n$ for every $X \subseteq V(F_0)$ of size $|X| \leq \frac{\log^2(s+m)}{p'}$,

(C) for every two disjoint subsets $X, Y \subseteq V(F_0)$ of sizes $|X|, |Y| \geq \frac{\log^{1.1}(s+m)}{p'}$, we have

$$e_{F_0}(X, Y) \leq (1 + 2\lambda)|X||Y|p'.$$

Lemma 6.4. *Let D be (n, λ, p) pseudo-random digraph with $0 < \lambda < 1/4$ and $p = \omega(\log^{14} n/n)$. Suppose that $V(D) = V_0 \cup V_1 \cup \cdots \cup V_\ell$ is a random (ℓ, s)-partition of $V(D)$ with $|V_0| = s$ and $n = m\ell + s$ and let F be the graph obtained from D by keeping every interior edge with probability $p_{in} = 1/(\alpha \ell \log n)$. Then with probability $1 - o(1/n)$, for every $j \in [\ell - 1]$ the directed subgraph $F_j = E_F(V_j, V_{j+1})$ contains $(1 - 4\lambda)mp \cdot p_{in}$ edge disjoint perfect matchings.*

16

6.3 Proof of Theorem 1.4

To prove Theorem 1.4 we use the following lemma (the analogue of Lemma 4.1) about the existence of many edge disjoint Hamilton cycles in special pseudo-random directed graphs.

Lemma 6.5. *Let $\mathcal{V} = (V_0, V_1, \ldots, V_\ell)$ be an (ℓ, s)-partition of a set X of size $n = \ell m + s$, chosen uniformly and independently at random. Let D be an (n, λ, p) pseudo-random graph on the vertex set X, with $p = \omega(\log^{14} n/n)$, and $0 < \lambda < 1/100$. Suppose that we select a random subdigraph F of $D(\mathcal{V})$ as follows:*

- *include each interior edge of $D(\mathcal{V})$ independently with probability p_{in};*

- *include each exterior edge of $D(\mathcal{V})$ independently with probability p_{ex}.*

Then, provided $p_{in} = (1 - o(1)) \cdot 1/(\alpha \ell \log n)$ and $p_{ex} = n^2/(\alpha^2 s^2 \ell^2 \log n)$, where $p' = p/\log^6 n$, $s = \sqrt{n/\alpha p'}$, $m = s/\log n$ and $\alpha = \alpha(n)$ is some function tending arbitrarily slowly to infinity with n, F contains $(1 - o(1))(1 - 4\lambda)mpp_{in}$ edge-disjoint Hamilton cycles with probability $1 - o(1/n)$.

Proof. To begin, look at the interior edges of F. For $j \in [\ell - 1]$ all edges of $E_D(V_j, V_{j+1})$ appear in F independently with probability p_{in}. Lemma 6.4 therefore gives that with probability $1 - o(1/t)$, $E_F(V_j, V_{j+1})$ contains $L := (1 - 4\lambda)mpp_{in}$ edge-disjoint perfect matchings $\{\mathcal{M}_{j,k}\}_{k=1}^L$ for all $j \in [\ell - 1]$. For each $k \in [L]$, taking the union of the edges in the matchings $\bigcup_{j=1}^{\ell-1} \mathcal{M}_{j,k}$ gives m directed paths, each directed from V_1 to V_ℓ and covering $\bigcup_{i=1}^\ell V_i$. Let $P_{k,1}, \ldots, P_{k,m}$ denote these paths and $\mathcal{P}_k = \{P_{k,1}, \ldots, P_{k,m}\}$.

Now assign to each exterior edge e of $D(\mathcal{V})$ a value $h(e) \in [L]$ chosen uniformly at random, all values chosen independently. Look at the exterior edges of F and for each $i \in [L]$ let H_i denote the subgraph of F with edge set $\{e \in E(F) : e \text{ exterior with } h(e) = i\}$.

Claim 6.6. *For any $k \in [L]$, the digraph $C_k := \{\vec{e} : \vec{e} \text{ is a directed edge of some path is } \mathcal{P}_k\} \cup H_k$ contains a directed Hamilton cycle with probability $1 - o(1/n)$.*

Note the proof of the lemma immediately follows from the claim, summing over $k \in [L]$.

To prove the claim, let

$$\mathcal{M}_k = \{(u_{k,i}, v_{k,i}) \in V_1 \times V_\ell : P_{k,i} \text{ is a } u_{k,i} - v_{k,i} \text{ directed path}\}.$$

By Remark 2.9 it suffices to prove that the auxiliary digraph $C_k(\mathcal{M}_k, V_0)$ contains a directed Hamilton cycle. Now note that $|V(C_k(\mathcal{M}_k, V_0))| = s + m$. We now wish to prove that with probability $1 - o(1/n)$ every $C_k(\mathcal{M}_k, V_0)$ is Hamiltonian. Observe that each $C_k(\mathcal{M}_k, V_0)$ was created from $F(\mathcal{M}_k, V_0)$ by keeping each edge e with probability at least

$$p_{ex}/|L| = \frac{n^2}{\alpha^2 s^2 \ell^2 \log n} \times \frac{1}{(1 - o(1))mpp_{in}}.$$

Using that $p_{in} = (1 - o(1))1/(\alpha \ell \log n)$, $p' = n/\alpha s^2$ and that $m\ell = (1 - o(1))n$ gives that $p_{ex}/|L| = (1 - o(1))p'/p$.

By applying Lemma 6.3, we see that $C_k(\mathcal{M}_k, V_0)$ satisfies properties (A), (B) and (C) with probability $1 - o(1/n)$. But (A), (B) and (C) give properties $(P1)$, $P(2)^*$ and $P(3)^*$ from Theorem 6.2, taking p' in place of p. Since $p' = (1 - o(1))p/\log^6 n = \omega(\log^8 n/n)$, by Theorem 6.2 any such $C_k(\mathcal{M}_k, V_0)$ are Hamiltonian. But if $C_k(\mathcal{M}_k, V_0)$ is Hamitonian, then so is C_k. Thus, this proves that C_k is Hamiltonian with probability $1 - o(1/n)$. $\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\alpha = \alpha(n)$ be some function tending arbitrarily slowly to infinity with n. Let $n = m\ell + s$ where $m = s/\log n$ and $s = \sqrt{n/\alpha p'}$.

Let $\mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(t)}$ be a collection of (ℓ, s)-partitions of $X = [n]$, where $\mathcal{V}^{(i)} = (V_0^{(i)}, \ldots, V_\ell^{(i)})$, chosen uniformly and independently at random and $t = \alpha \ell^2 \log n$. We will assign the edges of D among t edge disjoint subdigraphs $D^{(1)}, \ldots, D^{(t)}$ such that each $D^{(i)}$ preserves some pseudo-random properties. The digraphs $D^{(i)}$ are constructed as follows. Let

$$A_e := \{i \in [t] : e \text{ is interior in } D(\mathcal{V}^{(i)})\}; \qquad B_e := \{i \in [t] : e \text{ is exterior in } D(\mathcal{V}^{(i)})\}.$$

By Lemma 4.3, w.h.p. for each edge e we have $|A_e| = (1 + o(1))\frac{t}{\ell}$ and $|B_e| = ((1 + o(1))\frac{s^2 t}{n^2}$. We will now show that there exists a function f, $f(e) \in A_e \cup B_e$, such that if $D^{(i)}$ is the digraph given by $D^{(i)} = \{e : f(e) = i\}$, then $D^{(i)}$ contains $L := (1 - o(1))np/t$ directed Hamilton cycles. Clearly, this will complete the proof.

For each edge e choose a random value $f(e) \in A_e \cup B_e$ where each element in A_e is selected with probability $(1 - 1/\alpha)/|A_e|$ and each element in B_e is selected with probability $1/\alpha|B_e|$. For each $i \in [t]$, we take $D^{(i)}$ to be the digraph given by $D^{(i)} = \{e : f(e) = i\}$. First note that all edges e of $D(\mathcal{V}^{(i)})$ appear independently in $D^{(i)}$. If $e \in E(D)$ is an interior edge then the probability that it appears is $(1 - 1/\alpha)/|A_e| \geq (1 - o(1))\ell/t = (1 - o(1))1/\alpha\ell \log n := p_{in}$, since $t = \alpha\ell^2 \log n$. Similarly, each exterior edge e in $D \cap D(\mathcal{V}^{(i)})$ appears in $\mathcal{D}^{(i)}$ with probability $1/\alpha|B_e| \geq (1 - o(1))n^2/\alpha ts^2 = (1 - o(1))n^2/\alpha^2\ell^2 s^2 \log n := p_{ex}$.

Now note that the conditions of Lemma 6.5 are satisfied with these values (with $D^{(i)}$ in place of F), so with probability $1 - o(1/n)$, $D^{(i)}$ contains $L = (1 - o(1))(1 - 4\lambda)mpp_{in}$ edge disjoint Hamilton cycles. Therefore with probability $1 - o(1)$, $D^{(i)}$ contains L edge disjoint Hamilton cycles for each $i \in [t]$. Fix a choice of $\mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(t)}$ and f such that this holds. Using that $p_{in}t = \ell$ this gives that D contains $(1 - o(1))Lt = (1 - 4\lambda - o(1))mpp_{in}t = (1 - 4\lambda - o(1))mp\ell \geq (1 - 5\lambda)np$ edge-disjoint Hamilton cycles, as required. $\square$

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Appendix

Proof of Lemma 6.3. Recall that $p = \omega(\log^{14} n/n)$ and $p' = p/\log^6 n$, $s = \sqrt{n/\alpha p'}$, $m = s/\log n$, where $\alpha = \alpha(n)$ is some function tending arbitrarily slowly to infinity with n.

We will first prove (A). Note that for any choice of $\mathcal{M}$, and any vertex $v \in V(F_0)$ we have $d_{F_0}^+(v) = |N_{F_0}^+(u) \cap V_1| + |N_{F_0}^+(u) \cap V_0|$ for some $u \in V(D)$. Similarly $d_{F_0}^-(v) = |N_{F_0}^-(w) \cap V_\ell| + |N_{F_0}^-(w) \cap V_0|$ for some $w \in V(D)$. Let us thus estimate $|N_{F_0}^-(v) \cap V_i|$ for $i \in \{0, 1, \ell\}$ and for every $v \in V(D)$. Recall that $|N_D^\pm(v)| = (1 \pm \lambda)np$ by Definition 6.1 **(P1)**. As edges remain independently with probability $(1 - o(1))p'/p$ and $\mathcal{V}$ is chosen uniformly at random, for every vertex $v \in V(D)$ we have that

$$\mathbb{E}(|N_{F_0}^\pm(v) \cap V_i|) = (1 - o(1))\frac{p'}{p}\frac{|V_i|}{n}|N_D^\pm(v)| = (1 \pm \lambda \pm o(1))|V_i|p' = (1 \pm 2\lambda)|V_i|p',$$

But then by Chernoff's inequality we have that

$$\Pr(|N_{F_0}^\pm(v) \cap V_i| \notin (1 \pm 3\lambda)|V_i|p') \leq 2e^{-\frac{\lambda^2|V_i|p'}{3}} \leq 2e^{-\frac{\lambda^2|V_i|p'}{4}} \leq 2e^{-\frac{\lambda^2 \log^2 n}{4}} = o(1/n^3). \qquad (9)$$

The second last inequality holds since $|V_i|p' \geq mp' = \frac{sp'}{\log n} = \sqrt{\frac{np}{\alpha \log^6 n}} \cdot \frac{1}{\log n} \gg \sqrt{\frac{\log^{14} n}{\alpha \log^6 n}}\frac{1}{\log n} > \log^2 n$. By (9) this gives that with probability $1 - o(1/n)$ we have $d_{F_0}^\pm(v) = (1 \pm 3\lambda)(s+m)p'$ for all $v \in V(F_0)$, as required.

To see (B), note that for any $\mathcal{M}$, each set $X \subset V(F_0)$ corresponds to a set $X^* \subset V(D)$ with $|X^*| \leq 2|X|$, obtained by 'opening the pairs of X', i.e. $X^* = (X \setminus \mathcal{M}) \cup \{s_i, t_i : (s_i, t_i) \in X\}$. Thus to prove (B) it suffices to show that with probability $1 - o(1/n)$, every set $X^* \subset V(D)$ with $|X^*| \leq \frac{2\log^2(s+m)}{p'}$ satisfies $e_{F_0}(X^*) \leq \frac{|X^*|\log^{2.1} n}{2}$.

Now for $|X^*| \leq \frac{2\log^2(s+m)}{p'}$ since $p' = (1 \pm o(1))\frac{p}{\log^6 n}$, we have $|X^*| \leq \frac{4\log^8 n}{p}$. From Definition 6.1 **(P2)** we have

$$e_D(X^*) \leq (1 - \lambda)|X^*|\log^{8.02} n \leq \frac{(1 - \lambda)|X^*|\log^{8.05} n}{2}.$$

We now want to estimate $e_{F_0}(X^*)$. Since in F_0, each edge from D is included independently with probability $(1 \pm o(1))p'/p = \frac{(1 \pm o(1))}{\log^6 n}$. By Lemma 2.3, since $\mathbb{E}(e_{F_0}(X^*)) = (1 \pm o(1))e_D(X^*)p'/p \leq |X^*|\log^{2.05} n/2$ we have that

$$\Pr\left(e_{F_0}(X^*) > \frac{|X^*|\log^{2.1}(s+m)}{2}\right) \leq \left(\frac{2e \cdot e_D(X^*)p'/p}{|X|\log^{2.1}(s+m)}\right)^{|X|\log^{2.1}(s+m)}$$

$$\leq \left(\frac{40}{\log^{0.05} n}\right)^{|X|\log^{2.1}(s+m)}$$

The final inequality here holds as $\log(s+m) \geq \log n/3$ for $s \geq \sqrt{n}/\alpha(n) \geq n^{1/3}$. But there are $\binom{n}{x} \leq e^{x\log n}$ sets of size $|X^*| = x$. Therefore

$$\Pr\left(e_{F_0}(X^*) > |X^*|\log^{2.1}(s+m)/2 \text{ for some } X^*\right) \leq \sum_{x=1}^{m+s} e^{x\log n}\left(\frac{40}{\log^{0.05} n}\right)^{x\log^{2.1}(s+m)} = O\left(\frac{1}{n^2}\right).$$

This completes the proof of (B).

21

To prove (C), first note that by averaging it suffices to prove (C) when $X, Y \subseteq V(F_0)$ are two disjoint subsets with $|X|, |Y| = k = \lceil \frac{\log^{1.1} n}{p'} \rceil$. Given any choice of $\mathcal{M}$ and such sets X and Y, let $X^* = (X \setminus \mathcal{M}) \cup \{t_i \mid (s_i, t_i) \in X\}$ and $Y^* = (Y \setminus \mathcal{M}) \cup \{s_i \mid (s_i, t_i) \in Y\}$. Note that $|X| = |X^*|$ and $|Y| = |Y^*|$ and from **(P3)** of Definition 6.1 we have $e_{F_0}(X, Y) = e_{F_0}(X^*, Y^*)$. Thus to prove (C) for all choices of $\mathcal{M}$, it suffices to prove that with probability $1 - o(1/n)$, we have $e_{F_0}(X^*, Y^*) \leq (1 + 2\lambda)|X^*||Y^*|p'$ for all disjoint sets $X^*, Y^* \subset V(F_0)$ with $|X^*| = |Y^*| = k$.

To see this, note that for such X^*, Y^*, from Definition 6.1 (P3) we have $e_D(X^*, Y^*) = (1 \pm \lambda)|X^*||Y^*|p$. This gives that $\mathbb{E}(e_{F_0}(X^*, Y^*)) = (1 \pm \lambda \pm o(1))|X^*||Y^*|p'$ and by Chernoff's inequality we find

$$\Pr\left(e_{F_0}(X^*, Y^*)\right) > (1 + 2\lambda)|X^*||Y^*|p') \leq e^{-\frac{\lambda^2 |X^*||Y^*|p'}{6}} = e^{-\lambda^2 k^2 p'/6}.$$

Thus the probability that $e_{F_0}(X^*, Y^*) > (1 + 2\lambda)|X||Y|p'$ for some such pair is at most

$$\binom{n}{k}^2 e^{-\frac{\lambda^2 k^2 p'}{6}} \leq (ne^{-\frac{\lambda^2 k p'}{6}})^k = o(1/n),$$

where the final equality holds by choice of k. This completes the proof (C). $\qquad\square$

We now give the proof of the Lemma 6.4 stated in Section 6.2.

Proof of Lemma 6.4. To prove the lemma, we will first show that with probability $1 - o(1/n)$, for every $j \in [\ell - 1]$ the digraph F_j satisfies the following properties:

(i) $e_{F_j}(X, Y) = (1 \pm 2\lambda)|X||Y|p \cdot p_{in}$ for every two subsets $X \subseteq V_j$ and $Y \subseteq V_{j+1}$ with $|X|, |Y| \geq k = \lceil \frac{24 \log n}{\lambda^2 p p_{in}} \rceil$,

(ii) $e_{F_j}(X, Y) \leq \min\{|X|, |Y|\} \log^{2.05} n$ for all $X \subset V_j$ and $Y \subset V_{j+1}$ with $|X|, |Y| \leq k$,

(iii) $d^{\pm}(v, V_{j+1}) \geq (1 - 2\lambda)mp \cdot p_{in}$ for every $v \in V_j$.

We first prove (i). First note that by an easy averaging argument, it suffices to prove this for all such sets X and Y with $|X| = |Y| = k$. Now as D is (n, λ, p) pseudo-random and $k \geq \log^{1.1} n/p$, from property **(P3)** of Definiton 6.1 we have $e_D(X, Y) = (1 \pm \lambda)|X||Y|p$ for every such X and Y. Let $N_{X,Y}$ be the number of edges in $F_j[X, Y]$. Then $N_{X,Y} \sim \text{Bin}(e_D(X, Y), p_{in})$ and thus $\mathbb{E}(N_{X,Y}) = e_D(X, Y)p_{in} \geq (1 - \lambda)|X||Y|p \cdot p_{in}$. By Chernoff's inequality,

$$\Pr\left(N_{X,Y} \notin (1 \pm \lambda)e_D(X, Y)p_{in}\right) \leq e^{-\frac{\lambda^2}{3}(1-\lambda)|X||Y|p \cdot p_{in}} \leq e^{-\frac{\lambda^2 k^2 p p_{in}}{6}}.$$

By a union bound, this gives that

$$\Pr\left(N_{X,Y} \notin (1 \pm \lambda)e_D(X, Y)p_{in} \text{ for some pair } X \text{ and } Y\right) \leq \binom{m}{k}^2 e^{-\frac{\lambda^2}{3}(1-\lambda)|X||Y|p \cdot p_{in}} \leq n^{2k} e^{-\frac{\lambda^2 k^2 p p_{in}}{6}}$$

$$= (ne^{-\frac{\lambda^2 k p p_{in}}{12}})^{2k} = o(1/n).$$

The final equality here holds by the definition of k.

Property (ii) holds immediately from property **(P2)** in Definition 6.1.

We now show (iii). From **(P1)** of Definition 6.1 we have that $d_D^{\pm}(v) = (1 \pm \lambda)np$. Since for each $j \in [\ell - 1]$ the set V_{j+1} is chosen uniformly at random, the degree of v in V_{j+1} is distributed according

to the hypergeometric distribution with parameters n, $d_D^{\pm}(v)$, $|V_{j+1}|$. By Chernoff's inequality we have

$$\Pr\left(d_D^{\pm}(v, V_{j+1}) < (1-\lambda/2)\frac{d_D^{\pm}(v)m}{n}\right) \le e^{-\frac{\lambda^2 d_D^{\pm}(v)m}{6n}} \le e^{-\frac{\lambda^2}{12}mp} = o\left(1/n^3\right).$$

Therefore, with probability $1 - o(1/n)$ we have that for every $j \in [\ell - 1]$ and $v \in V_j$ we have $d_D^{\pm}(v, V_{j+1}) \ge (1-\lambda/2)mp$. We now use this to estimate $d_{F_j}^{\pm}(v)$ where $v \in V_j$. As in F every edge appears independently with probability p_{in}, by Chernoff's inequality we have

$$\Pr\left(d_F^{\pm}(v, V_{j+1}) < (1-2\lambda)mpp_{in}\right) \le e^{-\frac{\lambda^2 mp \cdot p_{in}}{6}} = o(1/n^2).$$

Thus with probability $1 - o(1/n)$ we have that $d_F^{\pm}(v, V_{j+1}) \ge (1-2\lambda)mp \cdot p_{in}$, i.e. (iii) holds.

Using (i), (ii) and (iii) we can now complete the proof of the lemma. It suffices to show that F_j contains an r-regular subgraph, where $r = (1-4\lambda)mp \cdot p_{in}$. To see this, by the Gale-Ryser theorem, it suffices to show that for all $X \subset V_j$ and $Y \subset V_{j+1}$ we have

$$e_F(X, Y) \ge r(|X| + |Y| - m). \tag{10}$$

Suppose that $|X| = x$ and $|Y| = y$. It clearly suffices to work with the case when $x + y \ge m$. First note that if $x, y \ge k$ then by (i) we have

$$e_F(X, Y) \ge (1-2\lambda)xyp \cdot p_{in} \ge (1-2\lambda)m(x+y-m)p \cdot p_{in} > r(x+y-m)$$

The second last inequality here holds since $(m-x)(m-y) \ge 0$. It remains to prove that (10) holds for X, Y satisfying $x + y \ge m$ with either $x \le k$ or $y \le k$. We will prove this for $x \le k$, as the other case is identical. Since $x + y \ge m$, we have $y \ge m - x$. But then $|Y^c| \le |X| \le k$ and

$$\begin{aligned}
e_F(X, Y) = e_F(X, V_{j+1}) - e_F(X, Y^c) &\ge (1-2\lambda)xmp \cdot p_{in} - (|X| + |Y^c|)\log^{2.05} n \\
&\ge x(1-2\lambda)mp \cdot p_{in} - 2x\log^{2.05} n \\
&= x(1-4\lambda)mp \cdot p_{in} + x(2\lambda mp \cdot p_{in} - 2\log^{2.05} n) \\
&\ge x(1-4\lambda)mp \cdot p_{in} \ge r(x+y-m).
\end{aligned}$$

The first inequality here holds by (ii) and (iii) and the third inequality holds since $\lambda mp \cdot p_{in} = \omega(log^{2.05}n)$ (note that this is true provided p is a sufficiently large power of $\log n$). $\qquad\square$